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STOCHASTIC MOTION DUE TO A SINGLE WAVE IN A MAGNETOPLASMA

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ABSTRACT

A single electrostatic wave in a magnetoplasma causes stochastic ion motion in several physically different situations. Various magnetic fields (uniform, tokamak, and mirror) and various propagation angles with respect to the field have been studied. A brief review of this work shows that all situations can be understood using the concept of overlapping resonances. Analytical calculations of the wave amplitude necessary for stochasticity have been carried out in some cases and compared with computer and laboratory experiments. In the case of an axisymmetric mirror field the calculations predict stochastic motion of ions with energy below a threshold that depends weakly on the wave amplitude and on the scale lengths of the magnetic field. Studies with an azimuthally asymmetric field show that the asymmetry causes substantial changes in the motion of some ions.

I. INTRODUCTION

Considerable study has been devoted to a class of stochasticity problems.[1-7] For these problems one asks the question, "In a magnetoplasma what effect on particle motion does a single electrostatic wave have?" The answer, of course, is that the wave causes stochastic motion of a certain group of particles. Characterizing this group generally provides information useful for a practical application. One can obtain this characterization, for all of the problems discussed in this paper, using the concept of overlapping resonances.[8] The physics of the resonances differs importantly from one problem to another, which leads to different characterizations of the stochastically moving particles.

In this paper I briefly discuss, in Sec. II, the various problems, emphasizing the physics of each problem and describing the group of particles that moves stochastically. In Sec. III I summarize the results of a detailed study[7] of ion motion in the presence of a wave in an axisymmetric mirror machine. Finally, in Sec. IV I describe a model appropriate for studies of ion motion in an azimuthally asymmetric mirror machine.

II. PHYSICS OF THE VARIOUS PROBLEMS

The first problem is that of ion motion in a uniform magnetic field $\underline{B}$ with a perturbing wave that propagates either obliquely[1] or perpendicularly[2,3] to $\underline{B}$. The ion feels oscillating electric fields at frequencies

$$\omega - k_z v_z + \ell\Omega, \quad \ell = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

where ω is the wave frequency, $k_z v_z$ is the Doppler shift due to motion along $\underline{B}$, and Ω is the gyrofrequency. The different frequencies (1) enter with weights given by the Bessel function $J_\ell(k_\perp a)$, where $k_\perp$ is the perpendicular wavenumber and a is the gyroradius. For the weights to be nonnegligible, we need $k_\perp a \gtrsim \ell$; the important values of ℓ are those that make (1) smallest in magnitude, namely $\ell \sim \omega/\Omega$. Thus, we find a necessary condition for stochasticity due to overlapping cyclotron resonances:

$$k_\perp a \gtrsim \omega/\Omega \quad . \tag{2}$$

Expressing (2) in terms of the perpendicular velocity $v_\perp$, we have

$$v_\perp \gtrsim \omega/k_\perp \quad .$$

The ions that may move stochastically are thus those with sufficiently high perpendicular energy.

In the second problem ions with negligible gyroradius ($k_\perp a << 1$) move stochastically in a tokamak magnetic field because the field strength B varies with distance s along a field line.[4] The ion feels the electrostatic wave (e.g., a trapped-ion mode) at frequencies

$$\omega - n\omega_b \quad , \quad n = 0, \pm 1, \pm 2, \ldots,$$

where ω_b is the bounce (or transit) frequency of motion along field lines. Since ω_b approaches zero at the separatrix, the velocity-space boundary between trapped and circulating ions, resonances $\omega = n\omega_b$ with arbitrarily large values of n occur near the separatrix. Resonances with adjacent values of n become extremely close together as n increases. Consequently, due to overlap of these bounce resonances a wave of arbitrarily small amplitude causes stochastic motion of trapped ions that reflect sufficiently close to the point s_{max} where B is a maximum. An analogous class of circulating ions also moves stochastically. These results depend on the assumed magnetic-field property $(d^2B/ds^2)_{s=s_{max}} \neq 0$.

The third problem treats the motion of ions with $k_\perp a = ka \gtrsim 1$ in the magnetic field of an axisymmetric mirror machine.[5-7] In contrast to the tokamak problem the shape of B(s) near s_{max} is irrelevant, since, in a mirror machine, most ions are near the surface on which B(s) is a minimum rather than a maximum. Near the minimum we can assume a parabolic well, $B(s) = B_0(1 + s^2/L^2)$. The exact shape of the well is not important but the fact that B varies with s, coupled with finite gyroradius $ka \gtrsim 1$, is crucial to the physics of this problem. Because $ka \gtrsim 1$ the ion feels the electrostatic wave, with a weight $J_1(ka)$, at the frequency

$$\omega + \omega_D - \Omega(t) \quad ,$$

where ω_D is the (positive) Doppler shift kv_d resulting from the ∇B-drift v_d, which is in the opposite direction from the (azimuthal) propagation direction of the wave. The gyrofrequency $\Omega(t) = (q/Mc)\, B[s(t)]$ varies about an average value $\bar{\Omega}$ at twice the frequency ω_b of bouncing between mirrors:

$$\Omega(t) = \bar{\Omega} - \tfrac{1}{2}\,\tilde{\Omega}\,\cos 2\omega_b t \quad .$$

The ion thus experiences a potential

$$\Phi(t) = \Phi_0\, J_1(ka)\, \cos\left[\int^t dt'\, \Omega(t') - (\omega_D + \omega)t\right]$$

$$= \Phi_0\, J_1(ka)\, \cos\,[(-\tilde{\Omega}/4\omega_b)\,\sin 2\omega_b t + (\bar{\Omega} - \omega_D - \omega)t]$$

$$= \Phi_0\, J_1(ka) \sum_{p=-\infty}^{\infty} J_p(\tilde{\Omega}/4\omega_b)\, \cos\,[(\bar{\Omega} - 2p\omega_b - \omega_D - \omega)t] \quad . \quad (3)$$

In (3) we have retained only the $\ell = 1$ term of a sum over ℓ that involves $J_\ell(ka)$, and we have suppressed initial phases of the bounce and cyclotron motion since these phases are unimportant for the present discussion. From (3) we see that the ion feels oscillating electric fields at frequencies

$$\bar{\Omega} - 2p\omega_b - \omega_D - \omega \quad , \quad p = 0, \pm 1, \pm 2, \ldots \quad . \quad (4)$$

The Bessel functions J_p, which play the roles of weighting functions in (3), can be expressed in terms of the parallel and perpendicular kinetic energies $W_\parallel$ and $W_\perp$, these energies being measured at the midplane (s = 0) of the mirror machine. The argument of J_p is

$$\frac{\tilde{\Omega}}{4\omega_b} = \frac{1}{4}\,\frac{W_\parallel}{W_\perp}\,\frac{L\Omega_0}{(2W_\perp/M)^{1/2}} \quad , \quad (5)$$

where M is the ion mass, L is the axial scale length of the magnetic field, and Ω_0 is the ion gyrofrequency at the midplane. A necessary condition for stochastic motion due to overlap of resonances with adjacent values of p is

$$\frac{\tilde{\Omega}}{4\omega_b} \gtrsim 1 \quad . \quad (6)$$

Relations (5) and (6) allow us to characterize the class of stochastically moving ions. These ions must have sufficiently low perpendicular energy $W_\perp$, the required value of $W_\perp$ decreasing as $W_\parallel/W_\perp$ decreases. Note that

$$W_\parallel/W_\perp = (s_{tp}/L)^2 \quad ,$$

where s_{tp} is the turning-point distance, at which the ion reflects from the mirror. The result that low-energy ions move stochastically is also found in a more complete calculation, which is described in the next section.

III. RESULTS ON STOCHASTICITY IN AN AXISYMMETRIC MIRROR MACHINE

A detailed treatment of stochasticity due to a wave in an axisymmetric mirror machine is available.[7] In this section I summarize the results of Ref. 7, which includes comparisons between theory and computer and laboratory experiments.

Theoretical work on stochasticity due to a wave in a mirror-machine plasma is motivated by data from the 2XIIB experiment at Lawrence Livermore Laboratory.[9] Measurements of electric potential fluctuations suggest that a single wave, which grows to large amplitude as an instability, propagates azimuthally in the plasma. The wave is described by a frequency ω and a wavenumber k that are unique and time independent.

Reference 7 studies the slab-model problem motivated by the 2XIIB experiment. An ion moves in a magnetic field

$$\underset{\sim}{B}(x,z) = B_0[(1 + z^2/L^2 + x/R_g)\hat{z} - (2xz/L^2)\hat{x}] \quad , \tag{7}$$

where L and R_g are the axial and radial scale lengths of the field. The motion is perturbed by an electrostatic wave with potential

$$\Phi = \Phi_0 \cos (ky + \omega t) \quad . \tag{8}$$

The analytical work of Ref. 7 begins with a Hamiltonian describing motion in field (7) and wave (8). Introduction of appropriate canonical variables allows one to express the Hamiltonian in the form

$$H(\overline{\xi},\mu;\phi,J) = H_0(\mu,J) + q\Phi_0 J_1(ka) \sum_p J_p(\tilde{\Omega}/4\omega_b) \cos (\overline{\xi} - 2p\phi) \quad , \tag{9}$$

where $(\overline{\xi},\mu)$ and (ϕ,J) are conjugate pairs of variables. The time-independent, two-degree-of-freedom Hamiltonian (9) represents a considerable simplification from the original time-dependent problem of three degrees of freedom. Considering each term of the sum on p separately, we can reduce (9) to a set of pendulum Hamiltonians

$$K_p(\chi,p_\chi) = (p_\chi - p_0)^2 G + q\Phi_0 J_1(ka) J_p(\tilde{\Omega}/4\omega_b) \cos \chi \quad , \tag{10}$$

where p_0 and G, which depend on the integer p, give the location of resonance p and the effective mass for motion near it. The widths of the resonances follow immediately from (10). The finite-width resonances are illustrated in Fig. 1 for $|p| \leq 2$ and a set of parameters suggested by 2XIIB data. Where adjacent resonances touch, we find points on the stochasticity boundary. For points in the $W_\parallel W_\perp$ plane below this boundary, ion motion is stochastic; above the boundary, motion is nonstochastic or, in mirror-theory terminology, superadiabatic.

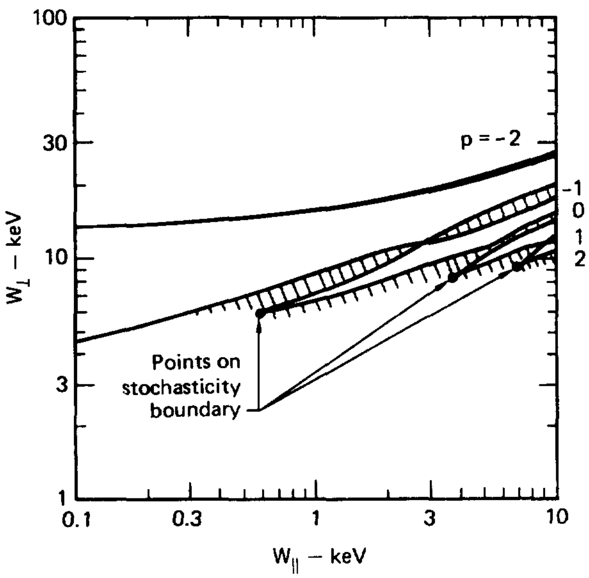

Fig. 1. Plot of finite-width resonances in the $W_\parallel W_\perp$ plane, showing three points at which adjacent resonances touch. The resonances are labeled by values of p. Parameters are $\omega = \Omega_0 = 2.8 \times 10^7\ s^{-1}$, $L = 60$ cm, $R_g = 30$ cm, $q\Phi_0 = 100$ eV, $k = 1\ cm^{-1}$, and M = the mass of the deuteron.

The predictions of the overlapping-resonances theory have been verified by numerical calculations of ion trajectories in magnetic field (7) and electrostatic wave (8). The trajectories separate into two classes as shown in Fig. 2, in which the kinetic energy of the ion is plotted in time. In Fig. 2(b) a low-energy ion shows the erratic, aperiodic behavior associated with stochasticity. In Fig. 2(a) two superadiabatic trajectories, both showing periodic variation of the kinetic energy, are plotted. Of the two superadiabatic trajectories the lower-energy one exhibits larger, slower excursions of the kinetic energy because in this case the ion motion is dominated by the $p = -1$ resonance; the higher-energy trajectory shows smaller, faster excursions because the ion is near none of the resonances.

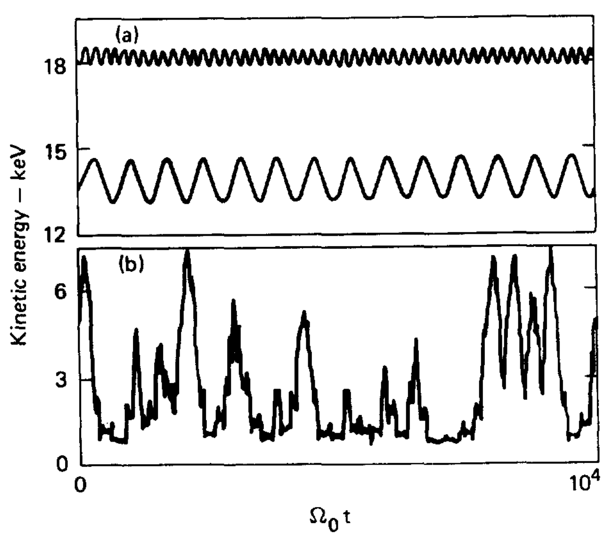

Fig. 2. Plots of kinetic energy versus time for the two types of ion trajectories: (a) superadiabatic, and (b) stochastic. Parameters are the same as in Fig. 1. At $t = 0$, $z = 0$ and $W_\parallel / W_\perp = 0.25$.

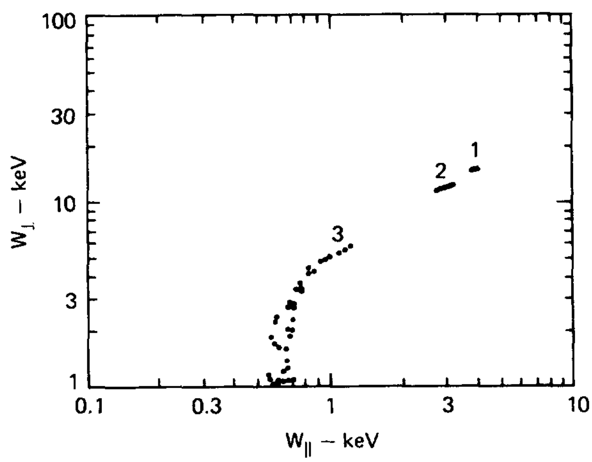

Fig. 3. Ion positions in the $W_\parallel W_\perp$ plane whenever z = 0 for the three trajectories shown in Fig. 2.

Figure 3 illustrates the same three trajectories as in Fig. 2 by plotting the ion position in the $W_\parallel W_\perp$ plane whenever the ion passes the midplane z = 0. The superadiabatic trajectories, labeled 1 and 2, remain near their initial points on the plot, while the stochastic trajectory, labeled 3, visits regions of velocity space far from its initial point.

Comparison between the overlapping-resonances theory and the trajectory calculations is shown in Fig. 4. From results like those in Figs. 2 and 3, the numerically determined stochasticity boundary is drawn using a solid curve. From Fig. 1 theoretically determined points on the boundary are plotted, and the points are joined by straight lines. The third boundary, drawn with a dashed curve, is determined by applying a

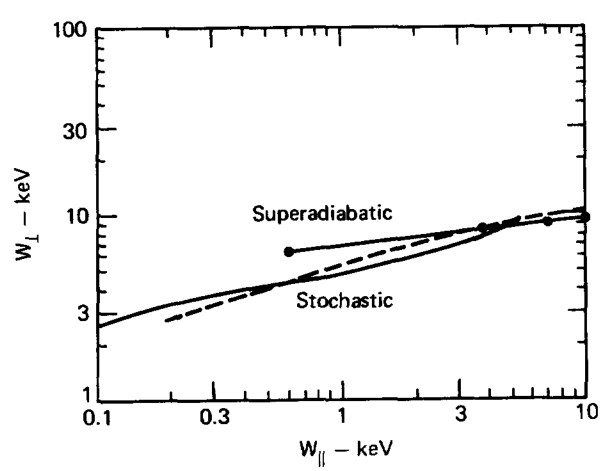

Fig. 4. Stochasticity boundaries in the $W_\parallel W_\perp$ plane, computed by the method of overlapping resonances (curve connecting four points), by the mapping method (dashed curve), and by numerical trajectory calculations (solid curve). Parameters are the same as in Fig. 1.

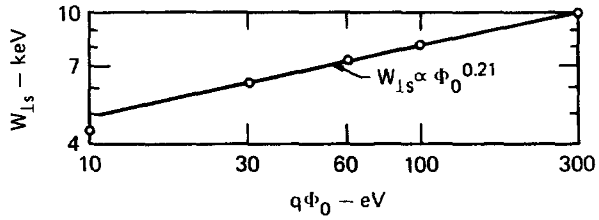

Fig. 5. Perpendicular energy $W_{\perp s}$ of an ion on the stochasticity boundary as a function of wave amplitude Φ_0.

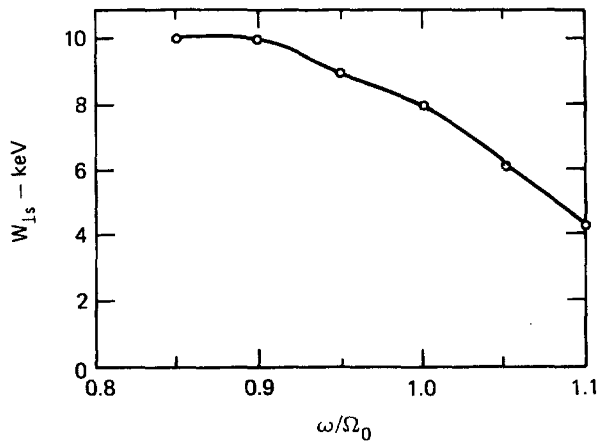

Fig. 6. Perpendicular energy $W_{\perp s}$ as a function of wave frequency ω.

slightly extended version of the mapping method used by Rosenbluth[5] and by Timofeev.[6] In this method one reduces the equations of motion to the standard mapping of Chirikov,[8] for which the stochasticity boundary is known from previous work.

After validation of the overlapping-resonances theory by comparison with numerical trajectory calculations, the theory is used to determine how the stochasticity boundary varies with parameters. In Figs. 5 and 6 are shown the parametric dependences of $W_{\perp s}$, defined as the value of $W_{\perp}$ on the stochasticity boundary at $W_{\parallel}$ = 2 keV. For these figures one parameter was varied while the others were fixed at the values used in Figs. 1 through 4. The variation of $W_{\perp s}$ with wave amplitude Φ_0 is rather weak, as shown in Fig. 5. In Fig. 6 we see how $W_{\perp s}$ varies with wave frequency ω. The dependences of $W_{\perp s}$ on the magnetic-field scale lengths R_g and L are very weak and are shown in Ref. 7.

The results of Ref. 7 force one to the following conclusion regarding the 2XIIB experiment. If the theory models all of the important features of the experiment, then in 2XIIB ions move stochastically if their perpendicular kinetic energy is less than about 8 keV, and they move superadiabatically otherwise. Certain observations in 2XIIB apparently contradict this conclusion, and motivation thus arises to improve the theoretical model. Such an improvement is described in the next section.

IV. MODEL FOR STOCHASTICITY STUDIES IN AN ASYMMETRIC MIRROR MACHINE

The slab-model magnetic field (7) used in Ref. 7 is a reasonable description of an axisymmetric mirror machine. However, most mirror experiments, including 2XIIB, involve a magnetic field with azimuthal asymmetry due to a quadrupole component of the field. Breaking of the azimuthal symmetry splits the frequencies (4) felt by an ion and leads to the doubly infinite set

$$\bar{\bar{\Omega}} - p'\bar{\omega}_b - n\omega_d - \omega \quad , \quad p',n = 0, \pm 1, \pm 2, \ldots \quad . \tag{11}$$

In (11), ω_d is the azimuthal drift frequency, $\bar{\omega}_b$ is the bounce frequency averaged over the drift motion, and $\bar{\bar{\Omega}}$ is the gyrofrequency averaged over both the bounce and drift motion. The large number of resonances associated with (11) seems likely to cause stochasticity more readily than in the axisymmetric case.

A suitable model magnetic field is[10]

$$\underline{B}(z,r,\theta) = f(z)\hat{z} - r(\tfrac{1}{2} f' - g \cos 2\theta)\hat{r} - rg \sin 2\theta\ \hat{\theta} \quad , \tag{12}$$

where g is a constant giving the strength of the quadrupole field and $f(z) = B_0(1 + z^2/L^2)$. Field (12) can be expressed as $\underline{B} = \nabla\alpha \times \nabla\beta$, where the Euler potentials α and β are

$$\alpha = \tfrac{1}{2}\, r^2 f(z)\ [\cosh c(z) - \cos 2\theta \sinh c(z)]$$

$$\beta = \tan^{-1} \{\exp\,[c(z)]\ \tan\theta\}$$

$$c(z) = (2gL/B_0)\ \tan^{-1}(z/L) \quad .$$

Guided by theoretical results on mirror-machine microinstabilities, I choose the model

$$\Phi(\alpha,\beta,t) = \Phi_0\, A_m[R(\alpha)]\, \cos\,(m\beta + \omega t) \tag{13}$$

for the potential due to the wave. In (13) Φ_0 is a constant giving the peak potential, the argument of A_m is

$$R = (\alpha/\alpha_0)^{1/2} \quad ,$$

where α_0 is the value of α at which the potential is greatest, the "radial" shape function is

$$A_m(x) = (m + 1)x^m\, (1 + mx^{m+1})^{-1} \quad ,$$

and m is the azimuthal mode number.

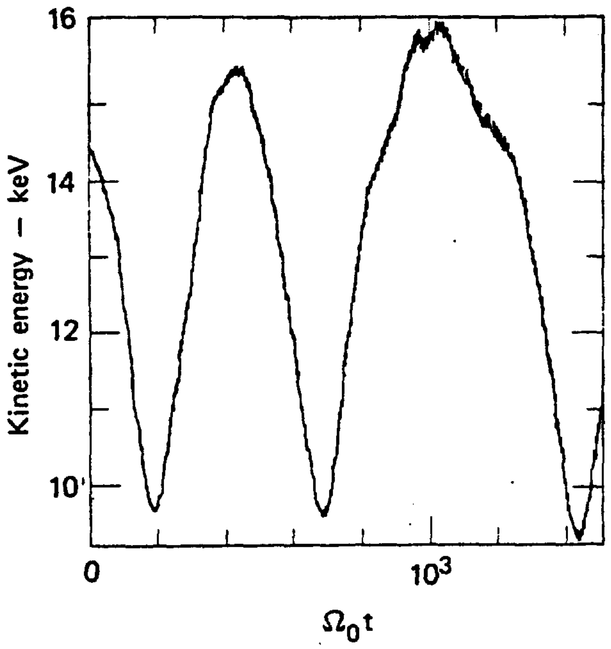

Fig. 7. Plot of kinetic energy versus time for an ion moving in a quadrupole field of strength given by $gL/B_0 = 2$. The values of ω, B_0, L, Φ_0, and M are close to the values used for Figs. 1 through 4. The mode number m is 7, and $\alpha_0 = \alpha(r = 7\text{ cm}, z = 0)$.

Ion trajectories in field (12) and wave (13) with the quadrupole-strength parameter g equal to zero reproduce the results of Ref. 7. With values of g comparable to those used in 2XIIB, we observe trajectories qualitatively different from the $g = 0$ trajectories. For example, the trajectory shown in Fig. 7 exhibits large, aperiodic excursions in the kinetic energy between the relatively high values of 9 and 16 keV. We are presently investigating the question of whether the quadrupole field alters the stochasticity boundary found in Ref. 7.

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